ON THE PROBLEM OF A MINIMUM OF A FUNCTIONAL IN THE INVESTIGATION OF THE STABILITY OF MOTION OF A BODY CONTAINING FLUID

PMM Vol. 31, No. 3, 1967. pp. 523-526

V.A. SAMSONOV
(Moscow)

(*Received March 25, 1966*)

In the paper by Rumiantsev [1] there is proved a theorem according to which the stability of the steady rotation of a rigid body with a cavity which is filled with two incompressible homogeneous fluids requires that the functional

$$W = \frac{1}{2}\frac{k_0^2}{S} + \Pi + \alpha\sigma + \alpha_1\sigma_1 + \alpha_2\sigma_2, \qquad \Pi = \Pi_0 + \int_{\tau_1} \rho_1\Pi_1\, d\tau + \int_{\tau_2} \rho_2\Pi_2\, d\tau$$

has an isolated minimum W_0 for the unperturbed motion.

Here k_0 is the moment of momentum of the whole system relative to the axis of rotation in the undisturbed motion; S is the moment of inertia of the system relative to the same axis in the perturbed state; τ_1, τ_2 are the volumes occupied by the fluids; ρ_1, ρ_2 are the corresponding densities of the fluids; Π_0, Π_1, Π_2 are the potentials of the forces which are acting on the body and the fluids, respectively; σ is the area of the interface of the fluids; σ_1, σ_2 are the areas of the wall in contact with the fluids; α, α_1 α_2 are the corresponding coefficients of surface tension. It is assumed that both fluids are in equilibrium relative to the body for undisturbed motion.

The existence of a weak minimum is a necessary condition for the minimum of the functional. The method of obtaining sufficient conditions for the weak minimum of the functional W from a study of its second variation $\delta^2 W$ is set out below.

1. The functional W clearly depends on the shape of the interface of the fluids (σ) and on the coordinates q_j $(j = 1,\ldots, n-1)$ which describe the position of the body (except the cyclic one q_n). The first variation of the functional W vanishes [1] for steady motion of the body which is described by Eqs.

$$q_j = 0, \qquad q_n = \omega t, \qquad \omega = \text{const}$$

Let the function l, given on the undisturbed surface $(\sigma)_0$ determine the deviation of the interface (σ) from the unperturbed surface $(\sigma)_0$. Then the second variation $\delta^2 W$ in the general case must consist of three parts: a quadratic functional in l, a quadratic form of the coordinates q_j and a functional linear in q_j and in l i.e. $\delta^2 W$ can be put in the form

$$\delta^2 W = P_1(l) + P_2(l, q_j) + U(q_j),$$

$$P_1(l) = (Ll, l), \qquad P_2(l, q_j) = 2(l, \Phi), \quad (l, \Phi) = \int_{(\sigma)_0} \Phi l\, d\sigma$$

Here Ll is a linear operator, Φ is a function of the form $a_1q_1 + \ldots + a_{n-1}q_{n-1}$, where a_j are certain functions given on the surface $(\sigma)_0$, $U(q_j)$ is the quadratic form of q_j. The explicit form of the operator Ll and the functions Φ and U depend on the external force and and the method of measuring the deviation l. One of the possible forms is shown below as an example. Another form of these expressions may be found in [2].

Let the functional P_1, (or the operator L) be positive definite. The corresponding conditions will be the first group of conditions for the weak minimum W.

One may reach a second set of conditions following the method set out in [3 and 4]. The functional $P_1 + P_2$ has a minimum for fixed q_j and this minimum [5] is found from the solution $l_1(q_j)$ of Eq.

$$Ll + \Phi = c_0 \tag{1}$$

($(l, c_0) = 0$ because the fluids are incompressible). Also

$$P_1 + P_2 = P_1(l - l_1) + {}^1/_2\, P_2(l_1)$$

It is clear that

$$\min(P_1 + P_2) = {}^1/_2 P_2(l_1) = (l_1, \Phi)$$

Because Eq. (1) is linear, its solution l_1 will be a linear function of q_j and (l_1, Φ) will be a quadratic form of q_j. The second variation can be put in the form

$$\delta^2 W = P_1(l - l_1) + V(q_j), \qquad V(q_j) = (l_1, \Phi) + U$$

The following theorem can now be proved.

Theorem. If P_1 is a positive definite functional and $V(q_j)$ is a positive definite quadratic form, then the functional W has a minimum W_0 for $q_j = 0$, $l \equiv 0$.

Proof. The difference between the values of the functional W in the perturbed and the unperturbed states of the system may be presented in the form

$$W - W_0 = \delta^2 W + a(\|l\|^2 + \|q\|^2), \qquad \|l\|^2 = (Ll, l), \qquad \|q\|^2 = q_1^2 + \ldots + q_{n-1}^2$$

where $a \to 0$ if $(\|l\|^2 + \|q\|^2) \to 0$; or this can be written after transformations as

$$W - W_0 = P_1(l - l_1) + V(q_j) + b(\|l - l_1\|^2 + \|q\|^2)$$

where $b \to 0$ if $(\|l - l_1\|^2 + \|q\|^2) \to 0$

As assumed there exists a number $d > 0$ such that

$$V(q_j) > d\|q\|^2$$

We will choose a number $\varepsilon > 0$ such that

$$|b| < \min\{{}^1/_2,\ {}^1/_2 d\} \quad \text{при} \quad \|l\|^2 + \|l_1\|^2 + \|q\|^2 < \varepsilon$$

Then

$$W - W_0 > {}^1/_2\|l - l_2\|^2 + {}^1/_2 d\|q\|^2 > 0, \qquad \text{if} \quad \|l\|^2 + \|q\|^2 \neq 0$$

i.e. W has a minimum W_0 for $q_j = 0$, $l \equiv 0$.

In order to construct the function V it is not necessary to solve Eq. (1). The coefficients of this quadratic form may be determined by any of the direct methods (e.g. Ritz's); this requires the minimization of the functional $P_1 + P_2$ for fixed q_j.

2. Let us consider the problem of the motion of a rigid body with a fixed point O under the action of the uniform gravitational force with acceleration g. Let us introduce the coordinate axes y_1, y_2, y_3 which are fixed, where y_3 is along the upward vertical. The axes x_1 x_2, x_3 which are fixed in the body are the principal axes of the ellipsoid of inertia at the point O. Polar coordinates r, φ are also introduced in the plane $x_1 x_2$.

We assume that the steady motion is a rotation of the body and the fluids in the cavity with constant angular velocity ω about the axis x_3 which coincides with the axis y_3.

For simplification of the calculations we will consider that the cavity is formed by a surface of revolution about the x_3 axis. The equation of this surface is $x_3 = \psi(r)$. The surface which separates the fluids in the cavity is also a surface of revolution with equation $x_3 = f(r)$. Then the axes x_i will be the principal axes of inertia for the whole system in the unperturbed motion. The fluid with density ρ_1 is below the surface of separation.

Let f be a single-valued function with bounded first and second derivatives. In this case we may take the deviation $l(r, \varphi)$ as the displacement of the surface of separation along the axis x_3 i.e. if $x_3 = h(r, \varphi)$ is the equation of the surface of separation in the perturbed state then $l = h - f$. Then

$$2\delta^2 W = P_1 + P_2 + U = \frac{1}{C}\Big(\rho\omega \iint_{(\Omega)} r^2 l\, d\Omega\Big)^2 + \iint_{(\Omega)} \Big[\rho g l^2 + \alpha\Big(l_r^{\,2}\{f\}^3 + \frac{l_\varphi^{\,2}\{f\}}{r^2}\Big)\Big] d\Omega -$$

$$- \alpha \int_\Gamma \mu l^2 \{f\}^3 d\Gamma + \iint_{(\Omega)} 2\rho(g + \omega^2 f)(\gamma_1 \cos\varphi + \gamma_2 \sin\varphi)\, l r\, d\Omega + \gamma_1^2 Q(A) + \gamma_2^2 Q(B)$$

$$\rho = \rho_1 - \rho_2, \qquad \{f\} = \frac{1}{\sqrt{1 + f_r^{\,2}}}, \qquad \mu = \frac{1}{\psi_r - f_r}\Big(\frac{1 + f_r^{\,2}}{1 + \psi_r^{\,2}}\,\psi_{rr} - f_{rr}\Big)$$

$$Q(A) = \omega^2(C - A) - Mgx_{3,0}, \qquad \gamma_i = \cos(y_3, x_i)$$

Here M is the mass; $x_{3,0}$ is the coordinate of the centre of gravity; A, B, C are the principal moments of inertia relative to the axes x_1, x_2, x_3 of the whole system taken as a single body in the undisturbed motion; (Ω) is the region bounded by the circle Γ which is the projection on the plane x_1, x_2 of the line of intersection of the surface of separation with the wall of the cavity. Partial derivatives with respect to r, φ are subscripted. For an unperturbed motion $\gamma_1 = \gamma_2 = 0$.

It is clear that P_1 will be a positive definite functional if

$$\rho_1 > \rho_2, \qquad \mu \leqslant 0 \tag{2}$$

Eq. (1) will have the form

$$Ll = \rho g l - \alpha\left[\frac{1}{r}(r l_r \{f\}^3)_r + \frac{1}{r^2} l_{\varphi\varphi}\{f\}\right] + \frac{r^2\rho^2\omega^2}{C}\iint\limits_{(\Omega)} r^2 l \, d\Omega = -\rho(g + \omega^2 f) r (\gamma_1 \cos\varphi + \gamma_2 \sin\varphi) + c_0$$

The quantity c_0 and the term containing the integral will vanish at the minimum so these can be omitted immediately. The boundary condition for the solution of this equation is

$$l_r - \mu l = 0 \,|_\Gamma$$

Since the operator L is linear the solution for l_1, may be split into two parts.

$$l_1 = \gamma_1 u(r) \cos\varphi + \gamma_2 v(r) \sin\varphi$$

For $u(r)$ and $v(r)$ we will have the same equation

$$L_1 u = -\rho(g + \omega^2 f) r, \qquad \cos\varphi\, L_1 u = L\,(u\cos\varphi) \tag{3}$$

with boundary condition

$$u_r = \mu u \,|_{r=R}$$

The quadratic form V now takes the form

$$2V = \gamma_1^2 [Q(A) + \pi\rho\nu] + \gamma_2^2 [Q(B) + \pi\rho\nu], \qquad \nu = \int\limits_0^R (g + \omega^2 f) u r^2 \, dr$$

When the solution of Eq. (3) is substituted into V we get the condition for it being positive definite,

$$\omega^2 (C - A) - Mg\, x_{3,0} + \pi\rho\nu > 0, \qquad A \geqslant B \tag{4}$$

Conditions (2) and (4) assures a weak minimum for W in this problem.

In the case of no surface tension ($\alpha = 0$) this condition corresponds to an analogous case in [3].

The numerical calculation of ν in a specific problem may be performed using the Ritz method and taking the Bessel functions $J_1(\lambda_j r)$ as coordinates. The number λ_j is the solution of Eq.

$$\frac{d}{dr} J_1(\lambda R) = \mu J_1(\lambda R)$$

Here R is the radius of the circle Γ. The Ritz system has then the form

$$\sum_i a_i b_{ij} = c_j, \qquad b_{ij} = \int\limits_0^R [L_1 J_1(\lambda_i r)] J_1(\lambda_j r)\, r\, dr, \qquad c_j = -\int\limits_0^R (g + \omega^2 f) J_1(\lambda_j r)\, r^2\, dr$$

and for ν we get

$$\nu = \sum_i a_i c_i$$

Let the cavity be cylindrical with radius $R = 1$. The surface of separation is at a finite distance from the end of the cavity. The parameters ρ, g, α, α_1, α_2 are such that the surface of separation for equilibrium of the body is given by the curve in Fig. 5 of the paper [6] for $W_0 = 1$.

Calculation of ν by the Ritz method gives for the first and second approximations:

$$\nu_1 = -0.236, \qquad \nu_2 = -0.245$$

These values are very close to one another and a rapid convergence is likely. Unfortunately, these values of ν are in error although they are sufficiently accurate for practical applications.

Consider now the case where the value ν is found analytically.

In [7] there is shown the form of the surface $(\sigma)_0$ at equilibrium ($\omega = 0$) with the coefficient α assumed small. The surface $(\sigma)_0$ in this case is a horizontal plane except for a circular region of width $\sim \sqrt{\alpha}$ near the wall of the cavity where the maximum distance of $(\sigma)_0$ from this plane is a value of the order $\sqrt{\alpha}$, and $f_r \sim 1$.

The quantity ν can be calculated in this case with the accuracy up to the terms of first order in α. In a cylindrical cavity, since $(l_1, \Phi) = \nu(\gamma_1^2 + \gamma_2^2)$ the value of ν can be considered as a minimum of the functional

$$W_1 = \int_0^R \left[gu^2 + \frac{\alpha}{\rho}\left(\{f\}^3 u_r^2 + \frac{1}{r^2}\{f\} u^2\right) + 2rgu\right] rdr$$

We will assume the functions u_1 and $u_{1,r}$ which are minimized, are bounded. Since $f_r \neq 0$ only in the region of width $\sim \sqrt{\alpha}$, the functional W_1 may be replaced by the functional

$$W_2 = \int_0^R \left[gu^2 + \frac{\alpha}{\rho}\left(u_r^2 + \frac{1}{r^2} u^2\right) + 2rgu\right] rdr$$

with the accuracy up to the terms of the first order in α. The minimizing function for W_2 clearly has the form

$$u_2 = -r + \frac{I_1(\lambda r)}{(d/dr)\, I_1(\lambda R)}, \quad \lambda^2 = \rho g / \alpha$$

and with first order accuracy

$$\nu = -\tfrac{1}{4} g R^4 + \alpha R^2 / \rho$$

Functionals W_1 and W_2 will be identical if the surface $(\sigma)_0$ is a plane. In the calculation of the coordinates of the centre of gravity $x_{3,0}$, the surface of separation may also be considered plane and the curvature introduces a correction $\sim \alpha^{3/2}$. This indicates that for a small surface tension the curvatures at the wall may be neglected and they only affect terms of higher order.

BIBLIOGRAPHY

1. Rumiantsev, V.V., On the stability of motion of a rigid body containing a fluid possessing surface tension PMM Vol. 28, No. 4, 1964.
2. Rumiantsev, V.V., On the theory of motion of rigid bodies with fluid-filled cavities. PMM Vol. 30, No. 1, 1966.
3. Pozharitskii, G.K. and Rumiantsev, V.V., The problem of the minimum in the question of stability of motion of a solid body with a liquid-filled cavity. PMM Vol. 27, No. 1, 1963.
4. Krein, S.G. and Moiseev, N.N., On the oscillations of a solid body containing fluid with a free surface. PMM Vol. 21, No. 2, 1957.
5. Mikhlin, S.G. and Smolitskii, Kh.L., Approximate Methods of the Solution of Differential and Integral Equations. Chapt. III, Izd. "NAUKA", 1965.
6. Beliaeva, N.A. , Myshkis, A.D. and Tiuptsov, A.D., Hydrostatics in weak gravitational fields. Equilibrium forms of the liquid surface. Izv. Akad. Nauk SSSR Mekhanika i Mashinostr.. No. 5, 1964.
7. Moiseev, N.N. and Chernous'ko, F.L., Problems of the oscillations of a fluid with surface tension. J. comp. Math. math. Phys. Vol. 5, No. 6, 1965.

Translated by J.J.L.B.